Two alternative Dirac equations with gravitation

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Abstract

An analysis of the classical-quantum correspondence shows that it needs to identify a preferred class of coordinate systems. One such class is that of the locally-geodesic systems. If a preferred reference frame is available, a different class emerges. From the classical Hamiltonian that rules geodesic motion, the correspondence yields two distinct Klein-Gordon (KG) equations and two distinct Dirac equations in a general metric, depending on the class selected. Each of the two Dirac equations can be put in generally-covariant form, is compatible with the corresponding KG equation, transforms the wave function as a 4-vector, and differs from the standard (Fock-Weyl) gravitational Dirac equation. One obeys the equivalence principle in the usually-accepted sense, which the Fock-Weyl equation does not.

Key words: Dirac and Klein-Gordon equations, wave mechanics, curved space-time, non-metric connection, preferred reference frame.

1 Introduction

Quantum effects in the classical gravitational field are being observed. Thus, in neutron interferometry, the phase shift which is predicted to occur, due to

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the effect of the Earth's gravitational potential, has indeed been measured [1], and even the smaller phase shift due the Earth's rotation has also been measured [2]. The same effects have been observed also in atomic interferometry [3, 4]. A more recent advance has been the experimental proof that the energy levels of neutrons are indeed quantized in the Earth's gravitational field, by measuring the neutrons transmission through a horizontal slit [5]. Until recently, the analysis of all these experiments has been based on the non-relativistic Schrödinger equation in the Newtonian gravity potential [5, 6, 7, 8], as is justified by the weakness of the gravitational field and the smallness of the velocities involved. However, one expects that the precision of such energy measurements will increase significantly. This is one of the reasons why theoretical research is also active in this field. Since neutrons are spin half particles, one may consider that, at least in the absence of a magnetic field, their behaviour should be correctly described by the Dirac equation. [1] In the case with a gravitational field, described by a curved space-time, the Dirac equation is usually modified to the form derived independently by Weyl and by Fock in 1929, hereafter the *Dirac-Fock-Weyl (DFW)* equation. Thus, work has been done to study the weak-field and/or non-relativistic limit of the DFW equation: see *e.g.* Refs. [9, 10, 11]. However, in the experiments on gravitational stationary states, one uses ultracold neutrons in the Earth's gravitational field [5]. It has been shown recently that, in this case, the corrections brought by the DFW equation to the non-relativistic Schrödinger equation in the gravity potential are quite hopelessly negligible [12, 13]. Nevertheless, one may expect that, in the future, experiments (possibly using lighter neutral particles: massive neutrinos?) should be able to check the relativistic corrections.

Moreover, the DFW equation is not the only possible extension of Dirac's equation to the case with a gravitational field. The DFW equation is *inspired by* the equivalence principle: it consists in rewriting the flat-space-time Dirac equation in a generally-covariant form, in such a way that the flat-space-time Dirac equation is recovered in a coordinate system in which, at the event con-

[1] For a neutral particle obeying Dirac's equation, the presence of an electromagnetic field has no effect whatsoever. This is contrary to the observed existence of a magnetic moment for the neutron, detected in magnetic scattering experiments: these gave the first indication that the neutron might be non-elementary, as is of course admitted in the framework of the quark model. Dirac's equation is thought to apply exactly to *elementary* spin-1/2 particles.

sidered, the metric tensor reduces to the flat form $\boldsymbol{\eta} \equiv \mathrm{diag}(1, -1, -1, -1)$ and the connection cancels. As one applies the same method to the Klein-Gordon (KG) equation, he obtains the so-called "minimally-coupled" version of the generally-covariant KG equation (see *e.g.* Ref. [14]): this is the case $\xi = 0$ of the generally-covariant KG equation containing the arbitrary parameter ξ which multiplies the scalar curvature [15]. However, in the case of the KG equation, one has to deal with the usual connection associated with the metric. In the case of the Dirac equation, in contrast, due to the spinor transformation used for the latter, one is then dealing with a connection of a very different kind: the "spinor connection," which is a non-metric one. Cancelling the metric connection, thus putting the KG equation to the flat-space-time form, is therefore a different operation from cancelling the spinor connection. Hence, the DFW equation does not generally reduce to the flat Dirac equation "in a local freely-falling frame," *i.e.,* in a coordinate system such that, at the event considered, the metric tensor reduces to the flat form $\boldsymbol{\eta}$ and the *metric* connection cancels (Appendix A)—that is, the DFW equation does not obey the genuine equivalence principle, although that very principle plays a guiding role in the transition from the flat Dirac equation to DFW. For the same motive, a solution of the DFW equation has no reason, in general, to be also a solution of the minimally-coupled KG equation. This also may be considered as a serious shortcoming of the DFW equation, insofar as the original (flat-space-time) Dirac equation is derived essentially from a factorization of the KG operator [16, 17, 18]. In summary, the application of the equivalence principle to the Dirac equation with spinor transformation leads to the DFW equation, which does not actually obey the equivalence principle, and for which the link with the KG equation is lost.

Starting from an analysis of the classical-quantum correspondence, we derived recently an alternative form of the Dirac equation in a gravitational field [19]. This derivation is based on wave mechanics, thus on directly associating a wave operator with a classical Hamiltonian. For this reason, it applies also to the KG equation (in fact, the method was first applied to the KG equation [20]), and the obtained gravitational versions of the KG and Dirac equations *are compatible*, in the sense that any solution of the latter is a solution of the former [19]. However, this gravitational Dirac equation was limited to the static case and does not obey the equivalence principle, although it has been noted that a *new* generally-covariant Dirac equation, obeying the equivalence principle, can be written as a by-product of the method utilized {Eq. (71) in Ref. [19]}. The aim of the present paper is to further study these two alternative proposals for a gravitational Dirac equation. In particular, we now derive both equations from wave mechanics, *i.e.,* from the classical Hamiltonian, and this in the general case (thus extending the first equation from the static case to the general case); we show that, in contrast with the DFW equation, each of these two equations is compatible with the corresponding KG equation; and, for each of these two equations, we derive the balance equation obeyed by the most obvious 4-current.

2 Classical-quantum correspondence for the Klein-Gordon and Dirac equations in a gravitational field

Instead of using the equivalence principle so as to adapt to gravity a wave equation originally derived for flat space-time, one might think of applying directly the classical-quantum correspondence. As we shall recall below (subsect. 2.3), there is indeed a classical Hamiltonian for the motion of a test particle in a gravitational field. However, canonical quantization will depend on coordinates, and in a curved space-time it seems that there are no preferred coordinates—whereas Galilean coordinates are preferred for a flat space-time. The points that we made [19, 21] are that

- i) the classical-quantum correspondence may be analysed from the purely mathematical correspondence between a wave operator and its dispersion equation, and from the wave-mechanical principle according to which the classical trajectories represent the skeleton of a wave pattern;

- ii) this analysis makes it clear that the classical-quantum correspondence needs to identify a preferred class of coordinate systems; at least in a static gravitational field, a such class does exist.

To make the paper self-contained, we briefly explain these points in subsects. 2.1 and 2.2 below. We add new remarks, in particular we now note that one may actually identify *two* distinct classes of coordinate systems, in each of which one may apply the classical-quantum correspondence. Then we apply this analysis to the Klein-Gordon (subsect. 2.3) and Dirac equations (subsect. 2.4). We end this Section by investigating the covariance of the obtained Dirac equation(s).

2.1 Analysis of the classical-quantum correspondence

Consider a linear partial differential equation of the second order (as is sufficient for quantum mechanics):

$$\mathrm{P}\psi \equiv \ a_0(X) + a_1^\mu(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi = 0, \tag{1}$$

where X is the position in the (configuration-)space-time V, of dimension $N+1$. The manifold V may occur as a product: $\mathrm{V} = \mathsf{R}\times\mathrm{M}$ with M an $N-$dimensional configuration space M, but this is not necessary. From subsect. 2.3 below, V will be the 4-dimensional space-time ($N=3$), and we will often adopt the corresponding language already now, but, before subsect. 2.3, N might be any positive integer. Let us look for "locally plane-wave" solutions [21], *i.e.,* wave functions $\psi(X) = A\,\exp[i\theta(X)]$ such that, at the event X considered, $\partial_\nu K_\mu(X) = 0$, where

$$K_\mu \equiv \partial_\mu\theta \qquad (\mu = 0, ..., N) \tag{2}$$

is the wave covector: $\mathbf{K} = (K_\mu) = (-\omega, \mathbf{k})$, with $\omega \equiv -K_0$ the frequency and $\mathbf{k} \equiv (K_j)$ $(j = 1, ..., N)$ the "spatial" wave covector. Substituting a such ψ into (1) leads to the *dispersion equation,* a polynomial equation for $\mathbf{K}$:

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i\,a_1^\mu(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0. \tag{3}$$

Clearly, the linear operator P (1) and its dispersion function $(X, \mathbf{K}) \mapsto \Pi_X(\mathbf{K})$ (3) are in *one-to-one correspondence* [19, 22]. The existence of *real* solution covectors $\mathbf{K}$ to (3) is the criterion that decides whether (1) can be termed a *wave equation.* The inverse correspondence [from (3) to (1)] is

$$K_\mu \longrightarrow \partial_\mu/i, \qquad (\mu = 0, ..., N). \tag{4}$$

The *dispersion relation(s)*: $\omega = W(\mathbf{k}; X)$, fix the wave mode [22]. Each of them is a particular root of $\Pi_X(\mathbf{K}) = 0$, considered as an equation for $\omega \equiv -K_0$. Witham's theory of dispersive waves [22] still contains the crucial observation that the propagation of $\mathbf{k}$ turns out to be ruled by a *Hamiltonian system*:

$$\frac{\mathrm{d}k_j}{\mathrm{d}t} = -\frac{\partial W}{\partial x^j}, \tag{5}$$

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial W}{\partial k_j} \qquad (t \equiv x^0,\ j = 1, ..., N). \tag{6}$$

Now the idea of de Broglie-Schrödinger's *wave mechanics* is that a classical Hamiltonian H describes the skeleton of a wave pattern. Then, the wave equation should give a dispersion W with the same Hamiltonian trajectories as H. The simplest way to do that is to assume that H and W are proportional, $H = \hbar W$... *This leads first to* $E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}$. *Then, substituting* $K_\mu \longrightarrow \partial_\mu / i$ *(4), it leads to the correspondence between a classical Hamiltonian and a "quantum" wave operator.* See Refs. [19, 21] for details.

2.2 The classical-quantum correspondence needs preferred coordinates

Thus far, a fixed system of coordinates (x^μ) has been assumed given on the (configuration-)space-time V. Yet we must allow for coordinate changes, which change the coefficients of operator P (1) in this way: [2]

$$a'_0 = a_0, \quad a'^{\rho}_1 = a^{\mu}_1 \frac{\partial x'^\rho}{\partial x^\mu} + a^{\mu\nu}_2 \frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu}, \quad a'^{\rho\theta}_2 = a^{\mu\nu}_2 \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\theta}{\partial x^\nu}. \tag{7}$$

It results from (7) that, at a given event $X \in$ V, the dispersion polynomial $\Pi_X(\mathbf{K})$ (3) remains invariant only if one makes an "infinitesimally linear" change, that is, if we have

$$\frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} = 0 \quad \forall \mu, \nu, \rho \in \{0, ..., N\} \tag{8}$$

at the event $X(x^\mu_0) = X(x'^\rho_0)$ considered [21]. It is easy to check that this defines an *equivalence relation* $\mathcal{R}_X$ between coordinate systems (charts $\chi : Y \mapsto (x^\mu)$) that are defined in a neighborhood of $X \in$ V:

$$\chi\ \mathcal{R}_X\ \chi' \text{ iff } \frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} = 0 \text{ at } X(x^\mu_0) = X(x'^\rho_0). \tag{9}$$

Thus, the dispersion polynomial (3) is well-defined only if we can identify a particular class of coordinate systems connected by changes satisfying (8),

[2] This follows from the transformation of the derivatives. We assume here that ψ is (or transforms as) a scalar. However, the definition of the dispersion function $\Pi_X(\mathbf{K})$ (3) and its one-to-one correspondence with operator P (4) remain valid if $\psi(X) \in \mathsf{R}^m$ or $\psi(X) \in \mathsf{C}^m$ and the coefficients of P are matrices with m rows and m columns [19].

and if we admit only those coordinate systems that belong to this class. In particular, if V is endowed with a pseudo-Riemannian metric $\boldsymbol{g}$, a relevant class is the class $\mathcal{C}_X^1$ made of the locally geodesic coordinate systems (LGCS) at X for $\boldsymbol{g}$, *i.e.*,

$$g_{\mu\nu,\rho}(X) = 0 \qquad \forall \mu, \nu, \rho \in \{0, ..., N\}. \tag{10}$$

This condition is indeed stable by any change satisfying (8), and, conversely, if some coordinate change gets an LGCS to another one, this change must satisfy condition (8) [19]. This means that $\mathcal{C}_X^1$ is exactly an equivalence class of relation $\mathcal{R}_X$.

But the well-definiteness of the dispersion polynomial $\Pi_X(\mathbf{K})$ is not the only condition necessary to apply the classical-quantum correspondence as we understand it here. The definition of the dispersion relation W (by solving $\Pi_X(\mathbf{K}) = 0$ for $\omega \equiv -K_0$) isolates the chosen time coordinate $t \equiv x^0$ among other possible choices $x'^0 = \phi((x^\mu))$. In the same way, the data of a classical Hamiltonian $H(\mathbf{p}, \mathbf{x}, t)$ does distinguish a special time coordinate t. (These two occurrences of a special time coordinate are bound together, of course, since the correspondence assumes that $H = \hbar W$.) In general, the wave equation thus associated with H will not be covariant under a change of the time coordinate, except for pure scale changes $x'^0 = ax^0$ [20]. Hence, *a priori*, what we should do would be to impose that only spatial coordinate changes are allowed, get a wave equation, and only then examine its behaviour under general coordinate changes [20]. To do that, we have to define in a natural way a class of coordinate systems exchanging by purely spatial and infinitesimally-linear changes:

$$x'^0 = ax^0, \quad x'^j = \phi^j((x^k)), \quad \frac{\partial^2 x'^j}{\partial x^k \partial x^l}(X) = 0. \tag{11}$$

In the case of a static Lorentzian [3] metric $\boldsymbol{g}$ on V, a class satisfying (11) does appear naturally: the class $\mathcal{C}_X^2$ of the coordinate systems adapted to the static character of the metric (*i.e.,* such that we have $g_{\mu\nu} = g_{\mu\nu}((x^j))$ and $g_{0j} = 0$), and which are locally geodesic, at the event X considered [or, equivalently in that case, at its spatial projection $\mathbf{x} \equiv (x^j)$], for the *spatial*

[3] By a "Lorentzian metric on V" we mean a pseudo-Riemannian metric with signature $(1, N)$. The static character of $\boldsymbol{g}$ is defined just below.

part $\boldsymbol{h}$ of the metric:

$$h_{jk,l}(X) = 0 \quad \forall j,k,l \in \{1,...,N\}. \tag{12}$$

Indeed, the transformations between static-adapted coordinate systems are just those satisfying $(11)_1$ and $(11)_2$ [23], [4] while, in the same way as for Eq. (8) in the case of any pseudo-Riemannian metric $\boldsymbol{g}$, the spatial coordinate systems that are LGCS for the spatial metric $\boldsymbol{h}$ are exactly those which exchange by infinitesimally-linear spatial transformations, Eq. $(11)_3$.

Now, let us consider a *general* Lorentzian metric $\boldsymbol{g}$ on V, and assume that for some reason we dispose of a *preferred reference frame* E, including the data of a preferred time coordinate T (up to a scale change). Thus, by definition, the coordinates that are *adapted to* E do exchange by changes satisfying $(11)_1$ and $(11)_2$. Moreover, the spatial metric $\boldsymbol{h}$ associated in the given frame E with the Lorentzian metric $\boldsymbol{g}$ [24] has then a privileged status, too. Hence, we may extend the definition of the class $\mathcal{C}^2_X$ to the case of a general Lorentzian metric $\boldsymbol{g}$, by defining $\mathcal{C}^2_X$ as the class of the systems adapted to E and that are LGCS at X for $\boldsymbol{h}$, Eq. (12). As for the class $\mathcal{C}^1_X$, two systems belonging to the class $\mathcal{C}^2_X$ must exchange by an infinitesimally-linear coordinate change (8). But, this time, the converse is not true: since the class $\mathcal{C}^2_X$ is restricted to purely spatial changes, not all changes satisfying (8) are internal to $\mathcal{C}^2_X$, in other words that class is smaller than an equivalence class of relation $\mathcal{R}_X$. Moreover, no LGCS *for* $\boldsymbol{g}$ is in general bound to the given frame E, because the local observer of the frame E is in general not in a "free fall." Therefore, the classes $\mathcal{C}^1_X$ and $\mathcal{C}^2_X$ have in general no intersection. (Recall that two equivalence classes are either equal or intersection-free.) An exception is (for $N = 3$, say) when the metric $\boldsymbol{g}$ is flat and the frame E is an inertial frame: in that case, $\mathcal{C}^2_X$ is contained in $\mathcal{C}^1_X$.

2.3 Klein-Gordon equation(s) in a gravitational field: derivation from the classical Hamiltonian

For a particle subjected to geodesic motion with a Lorentzian space-time metric $\boldsymbol{g}$, there is a classical Hamiltonian in the usual sense. To our knowledge, this result has been first got for the static case: in Ref. [20], it has

[4] thus, a static metric distinguishes a preferred reference frame.

been shown that the classical energy of the particle [24] is then a Hamiltonian, which may be expressed as a function of the canonical momentum $\mathbf{p}$ and the position $\mathbf{x}$ in the static reference frame as

$$H(\mathbf{p},\mathbf{x}) = [g_{00}(h^{jk}p_j p_k c^2 + m^2c^4)]^{1/2}, \qquad (h^{jk}) \equiv (h_{jk})^{-1} \tag{13}$$

(the canonical momentum $\mathbf{p}$ being in fact the usual momentum [20]). Bertschinger [25] argues that there is a Hamiltonian in the general case. He notes first that the following Hamiltonian defined on the 8-dimensional phase space:

$$\tilde{H}((p_\mu),(x^\nu)) \equiv \frac{1}{2}g^{\mu\nu}((x^\rho))p_\mu p_\nu \qquad (c=1), \tag{14}$$

has the geodesic lines as its trajectories, though this Hamiltonian is inconvenient because "every test particle has its own affine parameter" (a similar remark is Note 8 in Ref. [20]). Then he defines a Hamiltonian in the usual sense (*i.e.,* depending on the position in the 6-dimensional phase space, and on the independent time t) by "dimensional reduction" [26], which here is got simply by solving (14) for $H = -p_0$. Thus, H itself will depend on the coordinate system. However, Bertschinger [25] notes that $\tilde{H} = -\frac{1}{2}m^2$. Hence, in the most general case, $H \equiv -p_0$ satisfies the *generally-covariant* relation

$$g^{\mu\nu}p_\mu p_\nu - m^2 = 0 \qquad (c=1) \tag{15}$$

[which is easily checked from Eq. (13) in the static case. We changed the sign of $\boldsymbol{g}$ between Eqs. (14) and (15) to recover the signature $(+---)$.]

Therefore, we may apply the classical-quantum correspondence. First, assuming $\hbar = 1$ for convenience, the wave-mechanical correspondence $H = \hbar W$, *i.e.* $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, writes simply $p_\mu = K_\mu$, so that (15) is actually the dispersion equation (3) of the wave equation which is searched for. Then, the mathematical correspondence (4) gives immediately the wave equation

$$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = 0 \qquad (\hbar = c = 1). \tag{16}$$

However, following the discussion of subsect. 2.2, we must identify the class of coordinate systems in which we use the classical-quantum correspondence and in which we thus get Eq. (16), and then we must rewrite (16) in general systems. In the case that we have a preferred frame E (which case includes the static case), we may assume that (16) holds for the class $\mathcal{C}^2_X$, made of the systems adapted to E and which, in addition, satisfy (12). In any system satisfying (12), we have, moreover, $h^{jk}_{,l} = 0$ and $h_{,j} = 0$ at the event X considered, where $h \equiv \det(h_{jk})$. It follows that (16), assumed valid in a system of the class $\mathcal{C}^2_X$, may be rewritten as [5]

$$\Delta^{(\boldsymbol{h})}\psi - \frac{1}{c^2 g_{00}}\frac{\partial^2\psi}{\partial T^2} - M^2\psi = 0, \tag{17}$$

where T is the preferred time coordinate, $M \equiv \frac{mc}{\hbar}$, and where

$$\Delta^{(\boldsymbol{h})}\,\psi \equiv \psi^{|j}{}_{|j} = \frac{1}{\sqrt{h}}\left(\sqrt{h}\,h^{jk}\psi_{,k}\right)_{,j}, \tag{18}$$

adopting furthermore the notations

$$u^j{}_{|k} \equiv u^j_{,k} + \Delta^j_{lk}u^l \tag{19}$$

and

$$\psi^{|j} \equiv h^{jk}\psi_{,k} \tag{20}$$

for the covariant derivative with respect to the spatial metric $\boldsymbol{h}$, the Δ^j_{lk}'s being the Christoffel symbols of $\boldsymbol{h}$. Obviously, Eq. (17) remains valid in any system adapted to E, whether it satisfies (12) or not. [6]

On the other hand, without any assumption on a preferred frame, we may also write (16) for the class $\mathcal{C}^1_X$, made of the LGCS at X for $\boldsymbol{g}$: indeed, it happens that the (coordinate-dependent) Hamiltonian H satisfying (15) is available in any space-time coordinate system, and describes a unique motion (that along geodesics of the metric connection). This is a very particular situation for a classical Hamiltonian, and is the reason why our *a priori* strategy of specifying a preferred reference frame turns out to be not the

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[5] We assume moreover that the g_{0j} components cancel in one system adapted to E (and hence in all of them), which means that E admits a global synchronization [23, 24].

[6] Thus, knowing now that the Hamiltonian satisfying (15) is available in the general case for geodesic motion [25], we derived (17) in the general case; this is exactly Eq. (4.22) of Ref. [20], derived there in the static case. In the latter case, it is equivalent to Eq. (4.23) of Ref. [20], which had been suggested there as a possible extension of (4.22) to the general case. However, it turns out that Eq. (4.22) of Ref. [20], Eq. (17) here, is not generally equivalent to (4.23) there. Thus, the relevant time derivative in the general case is that w.r.t. the preferred time T, not that w.r.t. the "local time", $\partial_{t_\mathbf{x}} \equiv (1/\sqrt{g_{00}})\partial_T$.

only one possible. Rewriting (16) in a general coordinate system leads then, obviously, [7] to the minimally-coupled generally-covariant KG equation:

$$\Box^{(\boldsymbol{g})}\psi + M^2\psi = 0, \tag{21}$$

$$\Box^{(\boldsymbol{g})}\psi \equiv \psi^{;\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, g^{\mu\nu}\psi_{,\nu}\right)_{,\mu}, \;\; g \equiv \det(g_{\mu\nu}), \;\; (g^{\mu\nu}) \equiv (g_{\mu\nu})^{-1}. \tag{22}$$

Clearly, the two equations (17) and (21) are distinct (incompatible), except for a flat metric $\boldsymbol{g}$. This comes simply from the fact that they correspond to writing the classical-quantum correspondence in either of the two distinct classes of coordinate systems, $\mathcal{C}^2_X$ and $\mathcal{C}^1_X$ respectively.

2.4 Dirac equation(s) in a gravitational field: derivation from the classical Hamiltonian

In contrast with what happens in the nonrelativistic case, the relativistic dispersion equation (15) does not express the Hamiltonian $H = -p_0$ as an explicit polynomial in the canonical momentum $\mathbf{p} = (p_j)$ (or the spatial wave covector $\mathbf{k}$), but instead as an implicit algebraic function of it [19]. This (second-order) algebraic relationship (15) has the "right" solution, say $H(\mathbf{p}; X)$, but it has also another solution, *e.g.* simply $-H(\mathbf{p}; X)$ if $g_{0j} = 0$, and this other solution is inappropriate since it describes a different motion. Thus, (15) has too much solutions. As a consequence, the associated wave equation, *i.e.* the KG equation [either (17) or (21), which coincide in the case of flat space-time], also has too much solutions. Therefore, it is tempting to try a *factorization* of the dispersion equation associated with the algebraic relation (15) [19]:

$$\Pi_X(\mathbf{K}) \equiv [g^{\mu\nu}(X)K_\mu K_\nu - m^2]1_\mathrm{A} = [\alpha(X) + i\gamma^\mu(X)K_\mu][\beta(X) + i\zeta^\nu(X)K_\nu], \tag{23}$$

where 1_A means the unity in some algebra A, which must be larger than the complex field C (and hence may be noncommutative), because a decomposition (23) cannot occur in C. Identifying coefficients in (23), and applying

[7] Indeed, the l.h.s. of Eqs. (21) and (16) coincide in any system satisfying (10) at the event X considered. Semicolon means covariant derivative with the connection associated with $\boldsymbol{g}$.

the correspondence $K_\mu \to \partial_\mu / i$ (4) to (*e.g.*) the first of the two first-order polynomials on the r.h.s., leads [19] to the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m 1_{\mathrm{A}})\psi = 0, \tag{24}$$

where the objects $\gamma^\mu(X) \in \mathrm{A}$ have to obey the anticommutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\, 1_{\mathrm{A}}, \qquad \mu, \nu \in \{0, ..., 3\}. \tag{25}$$

This derivation works as it is summarized here, the metric $\boldsymbol{g}$ being not necessarily the flat metric, and being instead a general Lorentzian metric on space-time [19]. It works with a slight modification if one adds an electromagnetic potential A_μ [19]. As we know, it turns out that the smallest algebra allowing for (25) is that of 4×4 (complex) matrices, thus $\mathrm{A} = \mathsf{M}(4, \mathsf{C})$, hence $\psi(X) \in \mathsf{C}^4$.

As for the KG equation (16) obtained from the classical-quantum correspondence, Eq. (24) makes sense only in coordinate systems belonging to the identified class: either $\mathcal{C}^1_X$ or also, if a preferred reference frame E is available, the class $\mathcal{C}^2_X$. In either case, since (24) follows from the factorization (23) of that dispersion relation (15) which leads to the KG equation (16), it is clear that any solution of the Dirac equation (24) is also a solution of the KG equation (16). If we consider that (24) applies in coordinate systems belonging to the class $\mathcal{C}^1_X$, it may be rewritten in a general system simply as

$$(i\gamma^\nu D_\nu - M)\,\psi = 0, \qquad (D_\nu \psi)^\mu \equiv \psi^\mu_{\ ;\nu} \equiv \partial_\nu \psi^\mu + \Gamma^\mu_{\sigma\nu} \psi^\sigma, \tag{26}$$

because the latter equation coincides with (24) if we have (10) at the event X considered. (The $\Gamma^\mu_{\nu\sigma}$'s are the Christoffel symbols of $\boldsymbol{g}$.) Thus, we have now derived from wave mechanics Eq. (26), which had been merely noted in passing in Ref. [19]. As it had been noticed there, *Eq. (26) does obey the equivalence principle,* since it coincides with the flat-space-time Dirac equation if the metric (Levi-Civita) connection cancels at X and $\boldsymbol{g}(X) = \boldsymbol{\eta}$.

Similarly, if we have a preferred frame E at our disposal, and if we apply the classical-quantum correspondence in the class $\mathcal{C}^2_X$ of coordinate systems, we may rewrite (24) in any coordinate system adapted to E, but not necessarily satisfying (12), as

$$(i\gamma^\nu \Delta_\nu - M)\,\psi = 0, \tag{27}$$

with

$$(\Delta_\nu \psi)^\mu \equiv \begin{cases} \psi^\mu_{,\nu} & \text{if } \mu = 0 \text{ or } \nu = 0 \\ \\ \psi^j_{|k} & \text{if } \mu = j \text{ and } \nu = k \in \{1,2,3\} \end{cases} \tag{28}$$

[recall that $\psi^j_{|k}$ is the covariant derivative with respect to the spatial metric $\boldsymbol{h}$, Eq. (19): we have $\psi^j_{|k} = \psi^j_{,k}$ if the coordinate system satisfies (12)]. Using the explicit expression of the $\Gamma^\mu_{\nu\sigma}$'s for a metric such that $g_{0j} = 0$ (see *e.g.* Ref. [23], subsect. 3.2), one checks that (28) coincides, in the static case, with the expression previously found, involving the $\psi^\mu_{;\nu}$'s and other terms built with $\boldsymbol{g}$, Eqs. (74)-(77) in Ref. [19]. This means that Eq. (27) extends to the general case the gravitational Dirac equation previously derived from wave mechanics in the static case.

2.5 Covariance of the gravitational Dirac equations

To go from the form (24) of the Dirac equation to either of the two alternative general forms (26) and (27) [depending on whether the classical-quantum correspondence is applied in coordinate systems of the class $\mathcal{C}^1_X$ or $\mathcal{C}^2_X$, respectively], we implicitly assumed that (26) and (27) are covariant equations. Covariance means that, inside some definite class of coordinate systems, one has a transformation law for any object entering the equation, and that, if the equation is valid in one system of the class, it still applies to the transformed objects in any other system of this same class. For example, the DFW equation, which has just the same form as $(26)_1$ though with a definition of $D_\nu\psi$ differing from $(26)_2$, is generally-covariant; in it, the set of the γ^μ matrices (which is a *threefold array*) transforms like a *vector* $[\gamma'^\mu = (\partial x'^\mu/\partial x^\nu)\gamma^\nu]$, and the wave function array $\psi = (\psi^\mu)$ transforms like a *scalar*, *i.e.*, $\psi'^\mu(x'^\nu) = \psi^\mu(x^\nu)$ [9, 27]. The definition of $D_\nu\psi = ((D_\nu\psi)^\mu)$ for the DFW equation [9, 14, 27] turns out to imply that this double array transforms like a covector $[D'_\nu\psi' = (\partial x^\mu/\partial x'^\nu)D_\mu\psi]$, hence the l.h.s. of $(26)_1$ is, for the DFW equation, *invariant* (scalar) under any regular coordinate transform [12]. These transformation laws look somewhat unusual.

We claim that our equation (26) [involving the derivative $(26)_2$] is generally-covariant, too. This cannot be true with the same transformation laws as with the DFW equation, because the derivative $(26)_2$ does not make sense if $\psi = (\psi^\mu)$ behaves like a scalar. Actually, the general covariance of (26) has already been derived [19]: it involves transforming ψ as a *usual 4-vector*, and transforming the γ^μ matrices thus:

$$\gamma'^\mu \equiv L^\mu_\nu L \gamma^\nu L^{-1}, \qquad L \equiv (L^\mu_\nu), \quad L^\mu_\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu}. \tag{29}$$

Here, we note that (29) means simply (writing, as usual, the line index of the Dirac matrices as a superscript and the column index as a subscript) that the threefold array of the components $(\gamma^\mu)^\rho_\nu$ is a $\binom{2}{1}$ *tensor*. With ψ^μ being now a vector, $(D_\nu\psi)^\mu$ as given by $(26)_2$ is a $\binom{1}{1}$ tensor, hence the l.h.s. of $(26)_1$ is now a *vector* under a general coordinate change. At least to the present author, this transformation behaviour seems more natural, in that the orders of the tensorial transformation laws do now agree with the orders of the arrays—*i.e.,* a third-order tensor transformation for a threefold array, etc.

The same transformation laws may be [8] applied to the preferred-frame version (27) of the gravitational Dirac equation, which comes from applying the classical-quantum correspondence in the class $\mathcal{C}^2_X$. The difference with the former case is that, here, the coordinate changes are *a priori* restricted to be internal to the preferred frame E, thus purely spatial changes (along with $x'^0 = ax^0$), Eqs. $(11)_1$ and $(11)_2$. Considering successively the components $(\Delta_k\psi')^j, (\Delta_{0'}\psi')^j, (\Delta_k\psi')^{0'}$ and $(\Delta_{0'}\psi')^{0'}$, one checks easily that $(\Delta_\nu\psi)^\mu$ as defined by (28) is a $\binom{1}{1}$ tensor under changes $(11)_1$-$(11)_2$. Therefore, as was true under general changes for Eq. (26), the transformation of $(\gamma^\mu)^\rho_\nu$ as a $\binom{2}{1}$ tensor makes the l.h.s. of (27) a vector under the *a priori* allowed changes $(11)_1$-$(11)_2$. However, we may derive the covariance of (27) in a way which will allow us to extend it to general coordinate systems. The definition (28) of $(\Delta_\nu\psi)^\mu$ may be rewritten thus:

$$(\Delta_\nu\psi)^\mu = \psi^\mu_{,\nu} + \Delta^\mu_{\rho\nu}\psi^\rho, \tag{30}$$

with

$$\Delta^\mu_{\rho\nu} \equiv \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0 \text{ or } \rho = 0 \\ \\ \Delta^j_{lk} & \text{if } \mu = j \text{ and } \nu = k \text{ and } \rho = l \in \{1,2,3\}, \end{cases} \tag{31}$$

the Δ^j_{lk}'s $(j,k,l \in \{1,2,3\})$ being the Christoffel symbols of the spatial metric $\boldsymbol{h}$ in the preferred frame E. The definition (31) applies to coordinate systems

[8] and must be applied: also (28) does not make sense if $\psi = (\psi^\mu)$ behaves like a scalar.

adapted to E, which exchange by changes $(11)_1$-$(11)_2$. After any such change, the spatial connection Δ^j_{lk} transforms according to the usual rule, *i.e.,*

$$\Delta'^j_{kl} = \frac{\partial x'^j}{\partial x^m}\frac{\partial x^n}{\partial x'^k}\frac{\partial x^p}{\partial x'^l}\Delta^m_{np} + \frac{\partial x'^j}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^k \partial x'^l}. \tag{32}$$

It is easy to check that, as a consequence, a change $(11)_1$-$(11)_2$ gets the whole array $\Delta^\mu_{\rho\nu}$, as defined by (31), change according to the rule of connections:

$$\Delta'^\mu_{\nu\rho} = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\theta}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\rho}\Delta^\sigma_{\theta\tau} + \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\rho}. \tag{33}$$

The charts (coordinate systems) adapted to the preferred frame E must be two-by-two compatible, and, taken together, they must cover the space-time V: they make an "atlas" $\mathcal{A}$. Thus, for an atlas of charts of the manifold V, the modification of the array $\Delta^\mu_{\rho\nu}$ due to the change of chart obeys the transformation law of a connection. It follows that we endow V with a unique connection Δ in *defining* now the $\Delta'^\mu_{\nu\rho}$'s by (33) for a *general* space-time chart $\chi' : X \mapsto (x'^\mu)$. Indeed, it is proved in Appendix B that: i) the $\Delta'^\mu_{\nu\rho}$'s, obtained thus, are independent of the chart $\chi : X \mapsto (x^\nu)$, belonging to the atlas $\mathcal{A}$, which is used in Eq. (33); ii) moreover, the connection coefficients still transform according to the same rule (33) from a general system to another one; iii) there is only one connection on V which obeys (33) for any chart $\chi \in \mathcal{A}$ and for any general chart χ'.

Therefore, the rewriting (30) of the definition of $(\Delta_\nu \psi)^\mu$ is just the covariant derivative of ψ with the connection Δ thus defined, and it is now valid in any regular coordinate system [although, of course, Eq. (31) is not]. Thus, also our preferred-frame version (27) of the gravitational Dirac equation is generally-covariant—but it still keeps a preferred reference frame, since we have currently no way to define the connection coefficients otherwise than in transforming them from the preferred frame, as just explained. It is worth noting that Δ is not a metric connection, but still it is a torsion-free (symmetric) connection, $\Delta^\mu_{\rho\nu} = \Delta^\mu_{\nu\rho}$.

Thus, we have derived from wave mechanics two different versions of the Dirac equation in a gravitational field, Eqs. (26) and (27). Each of them is compatible with the corresponding KG equation, respectively Eq. (21) and Eq. (17), in the sense that any solution of (26) [resp. of (27)] is a solution of

(21) [resp. of (17)]—since, as we have noted before Eq. (26), this is true in the relevant coordinate systems, in which the Dirac and KG equations take respectively the form (24) and (16).

3 Balance equations for the new gravitational Dirac equations

3.1 Definition of the field of Dirac matrices

For the standard gravitational Dirac equation, which is the DFW equation, the field of the matrices $\gamma^\mu(X)$ satisfying the anticommutation relation (25) is defined covariantly from an orthonormal tetrad (u_α), with $u_\alpha \equiv a^\mu_\alpha \frac{\partial}{\partial x^\mu}$, by [9, 14, 27]

$$\gamma^\mu = a^\mu_\alpha \, \tilde{\gamma}^\alpha, \tag{34}$$

where $(\tilde{\gamma}^\alpha)$ is a set of "flat" Dirac matrices, that satisfies Eq. (25) with $\boldsymbol{g} = \boldsymbol{\eta} \equiv \mathrm{diag}(1, -1, -1, -1)$. Under a change of coordinates, the tetrad (u_α) is (of course) left unchanged, hence the matrix a^μ_α changes to

$$a'^\mu_\alpha = \frac{\partial x'^\mu}{\partial x^\nu} a^\nu_\alpha. \tag{35}$$

We can apply this in the same way to our alternative equations (26) and (27). However, for the DFW equation, the flat matrices $\tilde{\gamma}^\alpha$ are left unchanged in a coordinate change. With (34) and (35), this leads to the vector behaviour of the "deformed" set (γ^μ), already recalled. In contrast, for Eqs. (26) and (27), the array $\gamma^{\mu\nu}_\rho \equiv (\gamma^\mu)^\nu_{\ \rho}$ is a $\binom{2}{1}$ tensor. This is got with (34) and (35) by individually transforming each flat matrix $\tilde{\gamma}^\alpha$ as a $\binom{1}{1}$ tensor,

$$(\tilde{\gamma}'^\alpha)^\nu_{\ \rho} = \frac{\partial x'^\nu}{\partial x^\tau} \frac{\partial x^\phi}{\partial x'^\rho} (\tilde{\gamma}^\alpha)^\tau_{\ \phi}\,. \tag{36}$$

As needed, this preserves the anticommutation relation (25) for the $\tilde{\gamma}^\alpha$ matrices, with $\boldsymbol{\eta}$ in the place of $\boldsymbol{g}$. In turn, this anticommutation relation implies, as usual, *i.e.,* independently of the transformation behaviour of the matrices $\tilde{\gamma}^\alpha$, the anticommutation (25) for the actual metric $\boldsymbol{g}$. {This is checked by using (34) together with the orthonormality condition of the tetrad (u_α)—or, more precisely: together with the orthonormality condition of the dual tetrad (u^α_*) [such that $u^\alpha_*(u_\beta) = \delta^\alpha_\beta$], which follows from the latter.}

Alternatively, one may also define the set (γ^μ) of the deformed matrices simply by parallelly transporting the $\gamma^{\mu\nu}_\rho$ tensor along the geodesic lines of the metric connection associated with $\boldsymbol{g}$. Thus, let the γ^μ matrices be defined in any way (perhaps from a local tetrad) at some event $X_0 \in \mathrm{V}$, of course satisfying (25) at this event X_0. Then, for another event X, let G be the geodesic line (of the metric connection) that joins X_0 to X. We assume G to be unique; this is always the case in some neighborhood U of X_0—the smaller the curvature, the larger U. With the aim to follow spin half particles in the real world, assuming this uniqueness seems to be safe. Transporting the $\gamma^{\mu\nu}_\rho$ tensor along G does define matrices γ^μ matrices satisfying (25): indeed, this may be rewritten as

$$\gamma^{\mu\rho}_\sigma \gamma^{\nu\sigma}_\tau + \gamma^{\nu\rho}_\sigma \gamma^{\mu\sigma}_\tau = 2g^{\mu\nu}\delta^\rho_\tau, \qquad \mu, \nu, \rho, \tau \in \{0, ..., 3\}. \tag{37}$$

For any regular line, thus for G, there exists a coordinate system in which the connection coefficients $\Gamma^\mu_{\nu\rho}$ cancel along this line [28, 29]. In a such coordinate system, parallelly transported tensors are simply constant along G, component by component: this applies, by construction, to $\gamma^{\mu\nu}_\rho$, and it applies to $g^{\mu\nu}$ since $g^{\mu\nu}_{;\sigma} = 0$ implies that $g^{\mu\nu}$ is parallelly transported along any line. Hence, the equality (37), assumed to hold at X_0, holds true all along G (in this system, hence in all, this being a tensor equation), hence everywhere in U.

3.2 Balance equations for the usual current vector

The derivation of the current conservation for the flat Dirac equation (*e.g.* [18]) can be extended to the DFW equation [30]. The corresponding definition of the current may be adapted to the alternative gravitational Dirac equations (26) and (27), but this current obeys only a balance equation with a source term—as we now show. Avoiding reference to a special set of the flat Dirac matrices, the standard derivation may be based on a "hermitizing matrix" [30, 31], *i.e.,* a Hermitian matrix $B_{\rho\nu}$ such that each of the $\tilde{\gamma}^\alpha$ matrices is a Hermitian operator for the Hermitian product $(u, v) \equiv B_{\rho\nu} u^{\rho *} v^\nu$, which occurs iff we have

$$B_{\rho\nu}\, (\tilde{\gamma}^\alpha)^\nu_{\ \sigma} = B_{\nu\sigma}\, (\tilde{\gamma}^\alpha)^{\nu\,*}_{\ \rho} \qquad \alpha, \rho, \sigma \in \{0, ..., 3\}. \tag{38}$$

Clearly, if the "deformed" matrices γ^μ are related to the $\tilde{\gamma}^\alpha$'s by a tetrad (as is always possible), Eq. (34), then B is also hermitizing for the γ^μ's:

$$B_{\rho\nu}\gamma^{\mu\nu}_{\sigma} = B_{\nu\sigma}\gamma^{\mu\nu\,*}_{\rho} \qquad \mu, \rho, \sigma \in \{0, ..., 3\}. \tag{39}$$

Note that this is a tensor equation. We define the γ^μ's in that way at event X_0; for any other event X we define the $\gamma^{\mu\nu}_{\rho}(X)$ and $B_{\rho\nu}(X)$ tensors by parallel transport on the geodesic G, as above. We may define a 4-vector current by analogy with the flat Dirac and DFW equations:

$$j^\mu \equiv (\gamma^\mu \psi, \psi) = B_{\rho\nu}\gamma^{\mu\rho *}_{\sigma}\psi^{\sigma *}\psi^{\nu}, \tag{40}$$

and let us use again a coordinate system in which the Christoffel coefficients $\Gamma^\mu_{\nu\rho}$ cancel along G. Then $B_{\rho\nu} =$ Constant along G, hence (39) still applies (in these, hence in any coordinates). In other words, the parallelly-transported hermitizing matrix defines a hermitizing tensor field. We have also $D_\mu = \partial_\mu$ along G, whence

$$D_\mu j^\mu = B_{\rho\nu;\mu}\gamma^{\mu\rho *}_{\sigma}\psi^{\sigma *}\psi^{\nu} + ((D_\mu\gamma^\mu)\psi, \psi) + (\gamma^\mu D_\mu \psi, \psi) + (\psi, \gamma^\mu D_\mu \psi), \tag{41}$$

which again holds true in any coordinates, being a tensor equation. One may then use the relevant gravitational Dirac equation [either (26) or (27)]: for instance, if this is Eq. (26), the two last terms in (41) cancel one another. But anyway the r.h.s. remains with several source terms, which do not cancel individually nor as a whole (being largely independent); *e.g.,* unlike $\partial_\mu\gamma^\mu$ for the flat Dirac equation in Galilean coordinates and $D_\mu\gamma^\mu$ (with the covariant spinor derivative) for the DFW equation, we cannot expect that this $D_\mu\gamma^\mu$ (*i.e.,* tensor $\gamma^{\mu\nu}_{\rho;\mu}$) cancel. If it happens that there is a true conservation equation for either (26) or (27), the definition of the current must differ from (40).

4 Conclusion

The effects combining relativistic gravity with relativistic quantum mechanics are hardly measurable for the time being, but they should become so in the future. The present work stems from the idea that the already-existing theoretical tools to analyse such effects are not necessarily the last word. Thus, the standard (Dirac-Fock-Weyl or DFW) extension of the Dirac equation to gravitation is inspired by the equivalence principle, yet we find that it fails to obey this principle in the usually-accepted sense. Moreover, the compatibility with the Klein-Gordon equation seems to be lost with the DFW equation.

We start from an analysis of the classical-quantum correspondence, which turns out to make it work also for a curved space-time—whereas the way to the standard equations of relativistic quantum mechanics in curved space-time has been to rewrite the flat-space-time wave equations in generally-covariant form. However, we find that the classical-quantum correspondence has to be written in special classes of coordinate systems, and that two distinct such classes may be identified. (The second class needs to distinguish a preferred reference frame.) These two classes lead to two distinct Klein-Gordon equations, Eqs. (21) and (17), and to two distinct Dirac equations in a curved space-time, Eqs. (26) and (27). For flat space-time, the second class is a subclass of the first one, and the equations coincide. Each of the two Dirac equations can be put in generally-covariant form and is compatible with the corresponding KG equation. In addition, for each of these two Dirac equations, the wave function transforms as a 4-vector, and the set of the Dirac matrices builds a $\binom{2}{1}$ tensor. We have previously argued [19] that, the flat-space-time Dirac equation itself being left unchanged, its main physical consequences are also unchanged by this transformation behaviour. It may be added that anyway the transformation behaviour of the DFW equation under a coordinate change (as opposed to a change in the tetrad) does not coincide with that for the flat-space-time Dirac equation, either. One of the two alternative Dirac equations in curved space-time obeys the equivalence principle, and the other one has a preferred reference frame—the latter might lead to larger corrections to the Schrödinger equation in the Newtonian potential. However, the Hamiltonian operator remains to be studied for these two equations.

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A The spinor connection generally does not cancel in a local freely-falling frame

The covariant spinor derivative is $D_\mu \equiv \partial_\mu - \Gamma_\mu$, with [14, 27]

$$\Gamma_\mu = c_{\lambda\nu\mu} s^{\lambda\nu}, \qquad c_{\lambda\nu\mu} \equiv \frac{1}{4}\left(g_{\lambda\rho} b^\beta_{\nu,\mu} a^\rho_\beta - \Gamma_{\lambda\nu\mu}\right), \tag{42}$$

where $s^{\lambda\nu} \equiv \frac{1}{2}\left(\gamma^\lambda\gamma^\nu - \gamma^\nu\gamma^\lambda\right)$, and the matrix $A = (a^\mu_\alpha)$, with inverse $B = (b^\alpha_\mu)$, transforms the natural basis (e_μ) to the local tetrad (u_α), which is orthonormal, *i.e.,*

$$\boldsymbol{g}(u_\alpha, u_\beta) = a^\mu_\alpha a^\nu_\beta g_{\mu\nu} = \eta_{\alpha\beta}, \qquad \boldsymbol{\eta} \equiv \mathrm{diag}(1,-1,-1,-1) \equiv \mathrm{diag}(d_\mu), \tag{43}$$

which implies that

$$g_{\mu\nu} = b^\alpha_\mu b^\beta_\nu \eta_{\alpha\beta}. \tag{44}$$

The metric connection cancels at event X iff all derivatives $g_{\mu\nu,\rho}$ cancel at X, or equivalently iff the first-kind Christoffel symbols $\Gamma_{\lambda\nu\mu}(X)$ are all zero. We assume from now that this condition is fulfilled and that, in addition, the metric tensor $\boldsymbol{g}(X)$ reduces to the flat form $\boldsymbol{\eta}$—thus, we have a "local freely-falling frame"—and we check that, in general, the first part of $\Gamma_\mu(X)$ on the r.h.s. of Eq. (42) does not cancel then. Since $\boldsymbol{g}(X) = \boldsymbol{\eta}$, the orthonormality condition (43) is fulfilled by the natural basis (e_μ). Hence, the DFW equation being invariant under Lorentz transforms of the local tetrad [27], we may assume that [9]

$$A(X) = B(X) = 1_4, \tag{45}$$

which, by (44), ensures $\boldsymbol{g}(X) = \boldsymbol{\eta}$. With (45), the cancellation of the $g_{\mu\nu,\rho}$'s is equivalent, owing to (44), to

$$b^\nu_{\mu,\rho} d_\nu + b^\mu_{\nu,\rho} d_\mu = 0, \qquad \text{no sum on } \mu \text{ and } \nu. \tag{46}$$

In turn, this is satisfied iff the following conditions hold for all $\rho \in \{0, ..., 3\}$:

$$b^0_{0,\rho} = 0, \qquad b^0_{j,\rho} = b^j_{0,\rho} \quad j \in \{1,2,3\}, \tag{47}$$

[9] Assuming (45) just at the relevant event X, at which $\boldsymbol{g}(X) = \boldsymbol{\eta}$ in the chart utilized, we may restore the initial tetrad field, say $u'_\beta(Y)$, by a constant Lorentz transform L: $\forall Y\ u'_\beta(Y) = L^\alpha_\beta u_\alpha(Y)$, without changing the chart. This yields $A' = AL$, whence $B'_{,\mu} = L^{-1} B_{,\mu}$, thus leaving the $c_{\lambda\nu\mu}$'s unchanged in (42), whereas, owing to (45), $s^{\lambda\nu}(X)$ becomes $s'^{\lambda\nu}(X) = L^\lambda_\rho L^\nu_\sigma s^{\rho\sigma}(X)$. Therefore, a matrix $\Gamma'_\mu = c_{\lambda\nu\mu} s'^{\lambda\nu}$ cancels at X iff $\Gamma_\mu = c_{\lambda\nu\mu} s^{\lambda\nu}$ already does.

and

$$b^j_{k,\rho} = -b^k_{j,\rho}, \quad j,k \in \{1,2,3\}. \tag{48}$$

We get also from (45), using (42) and the antisymmetry of $s^{\lambda\nu}$:

$$\Gamma_\mu(X) = \frac{1}{4} \sum_{\rho<\nu} d_\rho \left(b^\rho_{\nu,\mu} - b^\nu_{\rho,\mu}\right) s^{\rho\nu}. \tag{49}$$

Accounting for (46), we obtain easily from (49):

$$\Gamma_\mu(X) = -\frac{1}{2} \sum_{j<k} b^j_{k,\mu} s^{jk}. \tag{50}$$

In addition to condition (46), that just rewrites $g_{\mu\nu,\rho} = 0$ when $A(X) = 1_4$, the only remaining constraint on the derivatives $b^\alpha_{\mu,\rho}(X)$ is that the tetrad orthogonality (43) must hold at every event, and hence

$$(a^\mu_\alpha a^\nu_\beta g_{\mu\nu})_{,\rho} = 0. \tag{51}$$

However, our assumption [$A(X) = 1_4$ and hence $\boldsymbol{g}(X) = \boldsymbol{\eta}$] implies that (51) is equivalent (at X) to

$$a^\beta_{\alpha,\rho} d_\beta + a^\alpha_{\beta,\rho} d_\alpha = 0, \qquad \text{no sum on } \alpha \text{ and } \beta. \tag{52}$$

The derivative of the inverse matrix $B = A^{-1}$ is $B_{,\rho} = -A^{-1}A_{,\rho}A^{-1}$, thus here $B_{,\rho} = -A_{,\rho}$, hence (52) is a consequence of (46). In other words, Eq. (51) does not bring any additional constraint on the derivatives $b^\alpha_{\mu,\rho}(X)$ in the present case. And since the constraint (46) is given explicitly by Eqs. (47) and (48), the derivatives $b^j_{k,\mu}(X) \quad (j < k)$ are left entirely free. It follows then from (50) that, indeed, *the spinor connection matrices* Γ_μ *generally do not cancel "in a local freely-falling frame," i.e.,* in a coordinate system such that $g_{\mu\nu,\rho}(X) = 0$ and $\boldsymbol{g}(X) = \boldsymbol{\eta}$ at the event X considered. Since the γ^μ matrices do coincide with the flat ones $\tilde{\gamma}^\mu$ in that case, the DFW equation $i\gamma^\mu(\partial_\mu - \Gamma_\mu)\psi = M\psi$ does not generally coincide with the flat Dirac equation $i\tilde{\gamma}^\mu\partial_\mu\psi = M\psi$ in a local freely-falling frame.

B Definition of a connection on a manifold V by extension from a smaller atlas of charts

What allows to define the connection Δ as explained after Eq. (33) is essentially the transitivity property for the transformation rule of a connection:

Lemma. *Let $\chi : X \mapsto (x^\mu), \chi' : X \mapsto (x'^\mu), \tilde{\chi} : X \mapsto (\tilde{x}^\mu)$ be three charts, two-by-two compatible, defined on the same open set* $\mathrm{U} \subset \mathrm{V}$*, and let $\Delta^\mu_{\nu\rho}, \Delta'^\mu_{\nu\rho}, \tilde{\Delta}^\mu_{\nu\rho}$ be (point-dependent) arrays such that one goes from Δ to Δ' by Eq. (33) and that the same transformation applies to go from Δ to $\tilde{\Delta}$, namely*

$$\tilde{\Delta}^\sigma_{\theta\tau} = \frac{\partial \tilde{x}^\sigma}{\partial x^\phi}\frac{\partial x^\psi}{\partial \tilde{x}^\theta}\frac{\partial x^\zeta}{\partial \tilde{x}^\tau}\Delta^\phi_{\psi\zeta} + \frac{\partial \tilde{x}^\sigma}{\partial x^\phi}\frac{\partial^2 x^\phi}{\partial \tilde{x}^\theta \partial \tilde{x}^\tau}. \tag{53}$$

Then, the transformation rule of a connection still applies to go from $\tilde{\Delta}$ to Δ'.

Proof: define a new array $\tilde{\Delta}'^\mu_{\nu\rho}$ by just Eq. (33), though substituting $\tilde{\Delta}$ for Δ and $\tilde{\Delta}'$ for Δ' (and substituting $\tilde{x}^\mu$ for x^μ and $\tilde{x}'^\mu$ for x'^μ). We will show that in fact $\tilde{\Delta}' = \Delta'$, which shall prove the result. Inserting the $\tilde{\Delta}^\sigma_{\theta\tau}$'s given by (53) into this definition of $\tilde{\Delta}'^\mu_{\nu\rho}$, we get (using implicitly the property that the three charts are two-by-two compatible at the $\mathcal{C}^2$ level and have the same domain of definition):

$$\tilde{\Delta}'^\mu_{\nu\rho} = \frac{\partial x'^\mu}{\partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\theta}{\partial x'^\nu}\frac{\partial \tilde{x}^\tau}{\partial x'^\rho}\frac{\partial \tilde{x}^\sigma}{\partial x^\phi}\frac{\partial x^\psi}{\partial \tilde{x}^\theta}\frac{\partial x^\zeta}{\partial \tilde{x}^\tau}\Delta^\phi_{\psi\zeta} + \frac{\partial x'^\mu}{\partial \tilde{x}^\sigma}\frac{\partial \tilde{x}^\theta}{\partial x'^\nu}\frac{\partial \tilde{x}^\tau}{\partial x'^\rho}\frac{\partial \tilde{x}^\sigma}{\partial x^\phi}\frac{\partial^2 x^\phi}{\partial \tilde{x}^\theta \partial \tilde{x}^\tau} + \frac{\partial x'^\mu}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\sigma}{\partial x'^\nu \partial x'^\rho}. \tag{54}$$

The second derivatives of composed functions are computed as

$$\frac{\partial^2 x^\phi}{\partial x'^\nu \partial x'^\rho} = \frac{\partial \tilde{x}^\theta}{\partial x'^\nu}\frac{\partial \tilde{x}^\tau}{\partial x'^\rho}\frac{\partial^2 x^\phi}{\partial \tilde{x}^\theta \partial \tilde{x}^\tau} + \frac{\partial x^\phi}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\sigma}{\partial x'^\nu \partial x'^\rho}. \tag{55}$$

Inserting this into (54) yields

$$\begin{aligned} \tilde{\Delta}'^\mu_{\nu\rho} &= \frac{\partial x'^\mu}{\partial x^\phi}\frac{\partial x^\psi}{\partial x'^\nu}\frac{\partial x^\zeta}{\partial x'^\rho}\Delta^\phi_{\psi\zeta} + \frac{\partial x'^\mu}{\partial x^\phi}\left(\frac{\partial^2 x^\phi}{\partial x'^\nu \partial x'^\rho} - \frac{\partial x^\phi}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\sigma}{\partial x'^\nu \partial x'^\rho}\right) + \frac{\partial x'^\mu}{\partial \tilde{x}^\sigma}\frac{\partial^2 \tilde{x}^\sigma}{\partial x'^\nu \partial x'^\rho} \\ &= \frac{\partial x'^\mu}{\partial x^\phi}\frac{\partial x^\psi}{\partial x'^\nu}\frac{\partial x^\zeta}{\partial x'^\rho}\Delta^\phi_{\psi\zeta} + \frac{\partial x'^\mu}{\partial x^\phi}\frac{\partial^2 x^\phi}{\partial x'^\nu \partial x'^\rho} = \Delta'^\mu_{\nu\rho}, \end{aligned} \tag{56}$$

which proves the Lemma.

Theorem. *Suppose that, on* V*, there is an atlas $\mathcal{A}$ such that, for each chart $\chi \in \mathcal{A}$, a (point-dependent) array $\Delta^\phi_{\psi\zeta}$ is defined, and that, for any two charts $\chi, \tilde{\chi} \in \mathcal{A}$, the arrays $\Delta^\phi_{\psi\zeta}$ and $\tilde{\Delta}^\sigma_{\theta\tau}$ are related by Eq. (53). Then, for any chart χ' belonging to the maximal atlas $\mathcal{M}$ compatible with $\mathcal{A}$, let us define $\Delta'^\mu_{\nu\rho}$ by Eq. (33). This makes sense and defines a connection on V (for its manifold structure defined by $\mathcal{M}$), which is the unique connection on V such* *that its coefficients $\Delta^{\phi}_{\psi\varsigma}$ are known for any chart $\chi \in \mathcal{A}$.*

Proof: i) Let U' be the domain of χ' and let $X \in \mathrm{U}'$. Since $\mathcal{A}$ is an atlas, there is a chart $\chi \in \mathcal{A}$ such that its domain U contains X, and since $\mathcal{M}$ is the maximal atlas compatible with $\mathcal{A}$, the charts χ and χ' are compatible. Hence, we may use Eq. (33) to define $\Delta'^{\mu}_{\nu\rho}$ on $\mathrm{U} \cap \mathrm{U}'$. If another chart $\tilde{\chi} \in \mathcal{A}$, with domain $\tilde{\mathrm{U}}$, is such that $\mathrm{U} \cap \mathrm{U}' \cap \tilde{\mathrm{U}} \neq \emptyset$, the proof of the Lemma shows that the definition of $\Delta'^{\mu}_{\nu\rho}$ from the chart $\tilde{\chi}$ (or rather from the transition map $\chi' \circ \tilde{\chi}^{-1}$) gives the same result as its definition from the chart χ, Eq. (56). Thus $\Delta'^{\mu}_{\nu\rho}$ is well-defined on U'.

ii) Let us now denote by $\tilde{\chi}$, not any more a chart in the small atlas $\mathcal{A}$, but instead another general chart (as χ'): $\tilde{\chi} \in \mathcal{M}$, with domain $\tilde{\mathrm{U}}$. Thus the foregoing paragraph provides us unambiguously with the two arrays $\Delta'^{\mu}_{\nu\rho}$ and $\tilde{\Delta}^{\sigma}_{\theta\tau}$, *i.e.*, in the general coordinate systems (x'^{μ}) and $(\tilde{x}^{\mu})$. Assume that $\mathrm{U}' \cap \tilde{\mathrm{U}} \neq \emptyset$, and let $X \in \mathrm{U}' \cap \tilde{\mathrm{U}}$. As above, there is a chart $\chi \in \mathcal{A}$ such that its domain U contains X, and the charts χ, $\tilde{\chi}$ and χ' are compatible. Since $\chi \in \mathcal{A}$, we dispose of the array $\Delta^{\mu}_{\nu\rho}$ in the coordinate system (x^{μ}). As shown in the foregoing paragraph, one gets the array $\Delta'^{\mu}_{\nu\rho}$ from the array $\Delta^{\mu}_{\nu\rho}$ by Eq. (33), *i.e.,* by the transformation rule of a connection, and the same rule applies to go from $\Delta^{\mu}_{\nu\rho}$ to $\tilde{\Delta}^{\sigma}_{\theta\tau}$. The Lemma shows that the transformation rule of a connection still applies to go from $\tilde{\Delta}$ to Δ' at any point $Y(x^{\mu}) = Y(x'^{\mu}) = Y(\tilde{x}^{\mu}) \in \mathrm{U} \cap \mathrm{U}' \cap \tilde{\mathrm{U}}$, hence in particular at X, thus at any point $X \in \mathrm{U}' \cap \tilde{\mathrm{U}}$. Thus, we have indeed defined a connection on V. [10]

iii) The uniqueness of the connection results, obviously, from the fact that it *must* transform according to Eq. (33), which is precisely used to define it.

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